2019
The Graduate School Entrance Examination
Mathematics
1:00 pm－3:30 pm

GENERAL INSTRUCTIONS

Answers should be written in Japanese or English.

1. Do not open the problem booklets, whether in English or Japanese, until the start of the examination is announced.
2. Notify your proctor if you find any printing or production errors.
3. Answer three problems (two problems for examinees of Department of Civil Engineering, Department of Systems Innovation, Department of Nuclear Engineering and Management, and Department of Technology Management for Innovation) out of the six problems in the problem booklet.
4. You are given three answer sheets (two answer sheets for examinees of Department of Civil Engineering, Department of Systems Innovation, Department of Nuclear Engineering and Management, and Department of Technology Management for Innovation). Use one answer sheet for each problem. You may use the reverse side if necessary.
5. Fill in your examinee number and the problem number in the designated places at the top of each answer sheet. The wedge-shaped marks on the top edge of each answer sheet represent the problem number that you answer (P 1, P 2, …, P 6) and also the class of the course (master M, doctor D) that you are applying. At the end of the examination, follow your proctor's instructions and cut out carefully the two corresponding wedge marks per sheet.
6. You may use the blank sheets of the problem booklet for rough papers without detaching them.
7. Any answer sheet with marks or symbols irrelevant to your answers is considered to be invalid.
8. You may not take the booklet or answer sheets with you after the examination.

Examinee Number	No.

Write your examinee number in the space provided above.

Problem 1

I. Obtain the general solution of the following differential equation:

$$x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = x^3. \tag{1}$$

II. Obtain the general solution of the following differential equation:

$$x^2 \frac{dy}{dx} - x^2y^2 + xy + 1 = 0. \tag{2}$$

Note that $y = \frac{1}{x}$ is a particular solution.

III. Let I_n be defined by:

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx, \tag{3}$$

where n is a non-negative integer.

1. Calculate I_0, I_1, and I_2.
2. Calculate I_n for $n \geq 2$.

Problem 2

I. Answer the following questions about the matrix $\boldsymbol{P}$:

$$\boldsymbol{P} = \begin{pmatrix} 0 & 0 & \frac{3}{2} \\ 2 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{pmatrix}. \tag{1}$$

1. Obtain all eigenvalues of the matrix $\boldsymbol{P}$ and the corresponding eigenvectors with unit norms.

2. Obtain $\boldsymbol{P}^2$ and $\boldsymbol{P}^3$.

II. Let $\boldsymbol{A}$ be the real matrix given by the block diagonal matrix:

$$\boldsymbol{A} = \begin{pmatrix} 0 & 0 & c & 0 & 0 \\ a & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e \\ 0 & 0 & 0 & d & 0 \end{pmatrix}. \tag{2}$$

Express succinctly the necessary and sufficient condition on a, b, c, d, and e, such that there exists a positive integer m for which $\boldsymbol{A}^m$ is the identity matrix (proof is not required).

III. The matrix $\boldsymbol{M}$ is a square matrix of order 12 with all elements taking either 0 or 1, such that each row and column has exactly one element being 1. Let k_0 be the minimum value of the positive integer k such that $\boldsymbol{M}^k$ is the identity matrix. For all possible matrices $\boldsymbol{M}$, give the maximum value of k_0 (proof is not required).

Problem 3

In the following, z denotes a complex number and i is the imaginary unit. The real part and the imaginary part of z are denoted by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, respectively.

I. Answer the following questions.

1. Give the solutions of $z^5 = 1$ in polar form. Plot the solutions on the complex plane.
2. The mapping f is defined by $f\colon z \mapsto f(z) = \exp(iz)$. Plot the image of the region $D = \{z\colon \mathrm{Re}(z) \geq 0,\ 1 \geq \mathrm{Im}(z) \geq 0\}$ under f on the complex plane.
3. Find the residue of the function $z^2 \exp\left(\frac{1}{z}\right)$ at $z = 0$.

II. Consider the complex function: $f(z) = \frac{(\log z)^2}{(z+a)^2}$, where a is a positive real number. The closed path C shown in Figure 3.1 is defined by $C = C_+ + C_R + C_- + C_r$, where $R > a > r > 0$. Here, $\log z$ takes the principal value on the path C_+. Answer the following questions.

1. Apply the residue theorem to calculate the contour integral $\oint_C f(z)\ dz$.
2. Use the result of Question II.1 to calculate the integral: $\int_0^{\infty} \frac{\log x}{(x+a)^2}\, dx$.

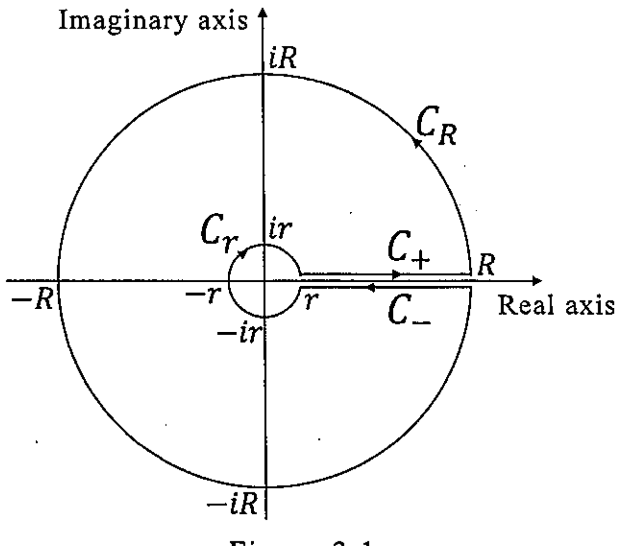

Figure 3.1

Problem 4

Answer the following questions on shapes in the three-dimensional orthogonal coordinate system xyz.

I. Consider the surface S_1 represented by the equation $x^2 + 2y^2 - z^2 = 0$. Find the equations expressed in x, y, and z of the normal line and the tangent plane T to the surface S_1 at the point $\mathrm{A}(2, 0, 2)$.

II. Consider the surface S_2 represented by the following set of equations with the parameters u and v:

$$\begin{cases} x = \dfrac{1}{\sqrt{2}} \cosh u \cos v & (1) \\ y = \dfrac{1}{2} \cosh u \sin v - \dfrac{1}{\sqrt{2}} \sinh u & (2) \\ z = \dfrac{1}{2} \cosh u \sin v + \dfrac{1}{\sqrt{2}} \sinh u \ , & (3) \end{cases}$$

where u and v are real numbers, and $0 \leq v < 2\pi$.
Let S_3 be the surface obtained by rotating the surface S_2 around the x-axis by $-\pi/4$. Here, the positive direction of rotation is the direction of the semi-circular arrow on the yz-plane shown in Figure 4.1.

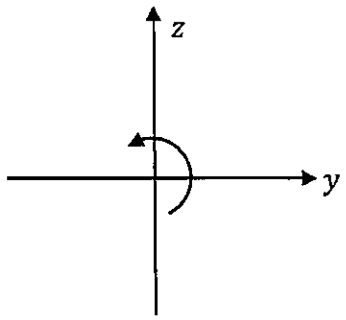

Figure 4.1

Answer the following questions.

1. Find the matrix $\boldsymbol{R}$ that represents the linear transformation rotating a shape around the x-axis by $-\pi/4$.
2. Find an equation expressed in x, y, and z for the surface S_3.
3. Find an equation expressed in x, y, and z for the surface S_2.

III. Consider the solid V that is enclosed by the surface S_3 obtained in Question II.2 and by the two planes $z = 1$ and $z = -1$. Answer the following questions.

1. Calculate the area of the cross section obtained by cutting the solid V with the xz-plane.
2. Calculate the area of the cross section obtained by cutting the solid V with the plane T obtained in Question I.

Problem 5

Consider the continuously differentiable function $f(x)$ of the real variable x. Let $f(x) \to 0$ as $|x| \to \infty$. $f(x)$, its derivative $f'(x)$, and $xf(x)$ are absolutely integrable. The Fourier transform of the function $f(x)$ is denoted by $\mathcal{F}\{f(x)\}(u)$ or equivalently by $\hat{f}(u)$, and defined by

$$\mathcal{F}\{f(x)\}(u) = \hat{f}(u) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp(-iux) dx, \tag{1}$$

where u is a real variable and i is the imaginary unit. The Fourier transform is defined in the same way for other functions.

I. Express $\mathcal{F}\{f'(x)\}(u)$ in terms of $\hat{f}(u)$ and u.

II. Express $\dfrac{d\hat{f}(u)}{du}$ in terms of $\mathcal{F}\{xf(x)\}(u)$.

III. Let the function $f(x) = \exp(-ax^2)$, where a is a positive real constant $(a > 0)$. The following relation holds for $f(x)$:

$$f'(x) = -2axf(x). \tag{2}$$

Apply the Fourier transform on both sides of Eq. (2) to obtain a first–order ordinary differential equation in $\hat{f}(u)$. Solve this ordinary differential equation to obtain $\hat{f}(u)$. Note that the integration constant in the solution of this ordinary differential equation can be obtained by calculating $\hat{f}(0)$ with the help of Eq. (1) and the value of the following improper integral:

$$\int_{-\infty}^{\infty} \exp(-ax^2) dx = \sqrt{\frac{\pi}{a}}. \tag{3}$$

IV. Consider the function $h(x,t)$ of the real variables x and t. Let $h(x,t)$ be defined for $-\infty < x < \infty$ and $t \geq 0$, and satisfy the following partial differential equation:

$$\frac{\partial h(x,t)}{\partial t} = \frac{\partial^2 h(x,t)}{\partial x^2} \qquad (t > 0), \tag{4}$$

given the initial condition

$$h(x,0) = \exp(-ax^2) \qquad (a > 0). \tag{5}$$

1. Apply the Fourier transform with respect to the variable x on both sides of the partial differential equation (4) to obtain an ordinary differential equation with $\hat{h}(u,t) \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(x,t)\exp(-iux)dx$ and the independent variable t.
2. By solving the ordinary differential equation found in Question IV.1, obtain $\hat{h}(u,t)$.
3. Use the inverse Fourier transform with respect to the variable u to obtain a solution $h(x,t)$ satisfying Eq. (4) and Eq. (5).

V. Consider the continuous function $g(x)$ and its Fourier transform $\hat{g}(u)$. Let $g(x) \to 0$ as $|x| \to \infty$ and $g(x)$ be absolutely integrable. The convolution of the functions $f(x)$ and $g(x)$ is defined by

$$(f * g)(x) \equiv \int_{-\infty}^{\infty} f(y)g(x-y)dy. \tag{6}$$

1. Express $\mathcal{F}\{(f * g)(x)\}(u)$ in terms of $\hat{f}(u)$ and $\hat{g}(u)$.
2. Here, the function $h(x,t)$ satisfies Eq. (4), given the initial condition $h(x,0) = g(x)$. Use the result of Question V.1 to find an integral representation of a solution $h(x,t)$, where $t > 0$.

Problem 6

Consider n random variables $X_1, X_2, \cdots, X_n$ that can take the values 0 and 1. Here, n is an integer greater than or equal to 4. The probability of an event A is denoted by $P(A)$, and the conditional probability of the event A given an event B is denoted by $P(A|B)$. The intersection between the event A and the event B is denoted by $A \wedge B$. Answer the following questions.

I. Let us assume that the $X_1, X_2, \cdots, X_n$ are independent. In addition, assume that each X_k $(k = 1, 2, \cdots, n)$ takes the value 1 with the probability p and the value 0 with the probability $1 - p$, i.e., $P(X_k = 1) = p$ and $P(X_k = 0) = 1 - p$.

1. Find the expected value and the variance of the sum of the $X_1, X_2, \cdots, X_n$.
2. The random variables $X_1, X_2, \cdots, X_n$ are arranged in the row $X_n \cdots X_2 X_1$. Let Y be the integer value obtained by regarding that row as an n-digit binary number. For example, in the case that $n = 4$, $Y = 5$ when the row $X_4 X_3 X_2 X_1$ is 0101, and $Y = 13$ when the row $X_4 X_3 X_2 X_1$ is 1101. Y is a random variable that takes integer values from 0 to $2^n - 1$. Obtain the expected value and variance of Y.

II. The values of the random variables $X_1, X_2, \cdots, X_n$ are obtained sequentially according to the following steps. First, X_1 takes the value 1 with the probability p and the value 0 with the probability $1 - p$. Then, X_k $(k = 2, 3, \cdots, n)$ takes the same value as X_{k-1} with the probability q and the value different from X_{k-1} with the probability $1 - q$, i.e.,
$P(X_k = 1 | X_{k-1} = 1) = P(X_k = 0 | X_{k-1} = 0) = q$
and $P(X_k = 1 | X_{k-1} = 0) = P(X_k = 0 | X_{k-1} = 1) = 1 - q$.

1. Let $P(X_k = 1)$ be represented by r_k, where k is an integer varying from 1 to n. Derive a recurrence equation for r_k. Solve this recurrence equation to express r_k with p, q, and k.
2. Obtain the probability $P(X_1 = 1 \wedge X_2 = 0 \wedge X_3 = 1 \wedge X_4 = 0)$.
3. Obtain the probability $P(X_3 = 1 \mid X_1 = 0 \wedge X_2 = 1 \wedge X_4 = 1)$.