2018
The Graduate School Entrance Examination
Mathematics
1:00 pm－3:30 pm

GENERAL INSTRUCTIONS

Answers should be written in Japanese or English.

1. Do not open the problem booklets, whether in English or Japanese, until the start of the examination is announced.
2. Notify your proctor if you find any printing or production errors.
3. Answer three problems (two problems for examinees of Department of Civil Engineering, Department of Systems Innovation, Department of Nuclear Engineering and Management, and Department of Technology Management for Innovation) out of the six problems in the problem booklet.
4. You are given three answer sheets (two answer sheets for examinees of Department of Civil Engineering, Department of Systems Innovation, Department of Nuclear Engineering and Management, and Department of Technology Management for Innovation). Use one answer sheet for each problem. You may use the reverse side if necessary.
5. Fill in your examinee number and the problem number in the designated places at the top of each answer sheet. The wedge-shaped marks on the top edge of each answer sheet represent the problem number that you answer (P 1, P 2, …, P 6) and also the class of the course (master M, doctor D) that you are applying. At the end of the examination, follow your proctor's instructions and cut out carefully the two corresponding wedge marks per sheet.
6. You may use the blank sheets of the problem booklet for rough papers without detaching them.
7. Any answer sheet with marks or symbols irrelevant to your answers is considered to be invalid.
8. You may not take the booklet or answer sheets with you after the examination.

Examinee Number	No.

Write your examinee number in the space provided above.

Problem 1

I. Find the general solutions of the following differential equations.

1. $$\frac{d^2y}{dx^2}+2\frac{dy}{dx}-3y=e^x\cos x \quad (1)$$

2. $$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\frac{4}{x^2}y=\left(\frac{2\log x}{x}\right)^2 \quad (2)$$

II. Answer the following questions for the partial differential equation represented in Equation (3) and the boundary conditions represented in Equations (4)-(7):

$$\frac{\partial^2 u(x,y)}{\partial x^2}+\frac{\partial^2 u(x,y)}{\partial y^2}=0 \qquad (0\le x,\ 0\le y\le 1), \quad (3)$$

$$\begin{cases} \lim_{x\to+\infty} u(x,y)=0 & (4)\\ \left.\frac{\partial u(x,y)}{\partial y}\right|_{y=0}=0 & (5)\\ u(x,1)=0 & (6)\\ \left.\frac{\partial u(x,y)}{\partial x}\right|_{x=0}=1+\cos\pi y. & (7)\end{cases}$$

1. Find the solution which satisfies Equations (3) and (4) in the form of $u(x,y)=X(x)\cdot Y(y)$.
2. Find the solution satisfying Equations (5) and (6) for the solution of Question II.1.
3. Find the solution of the partial differential equation (3) satisfying all the boundary conditions given in Equations (4)-(7), using the solution of Question II.2.

Problem 2

I. Suppose that λ is an eigenvalue of a regular matrix $\boldsymbol{P}$, prove that:

1. λ is not zero.
2. λ^{-1} is an eigenvalue of $\boldsymbol{P}^{-1}$ and λ^{n} is an eigenvalue of $\boldsymbol{P}^{n}$, where n is a positive integer.

II. Suppose $\boldsymbol{P}$ is an orthogonal matrix. When the following symmetric matrix $\boldsymbol{A}$ can be diagonalized by $\boldsymbol{P}$, find the matrix $\boldsymbol{P}$ and obtain the diagonalized matrix.

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

III. When a matrix $\boldsymbol{P}$, and vectors $\boldsymbol{r}$ and $\boldsymbol{x}$ are given as

$$\boldsymbol{P} = \begin{pmatrix} 1 & 1 & 1 \\ p & p^2 & p^3 \\ q & q^2 & q^3 \end{pmatrix}, \qquad \boldsymbol{r} = \begin{pmatrix} r \\ r^2 \\ r^3 \end{pmatrix}, \qquad \boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where p, q, and r are non-zero real numbers that differ from each other.

1. Find the condition that p and q must satisfy in order for $\boldsymbol{P}$ to be a regular matrix.
2. When $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{r}$ has a single solution, obtain $\boldsymbol{x}$. Here, $\boldsymbol{P}^{\mathrm{T}}$ is the transposed matrix of $\boldsymbol{P}$.

IV. The matrix $\boldsymbol{P}_n$ is an n-th order square matrix ($n \geq 2$), as shown below, where p and q are real numbers that differ from each other.

$$
\boldsymbol{P}_n = \begin{pmatrix} p+q & q & 0 & \cdots & 0 & 0 \\ p & p+q & \ddots & \ddots & \vdots & \vdots \\ 0 & p & \ddots & \ddots & 0 & \vdots \\ \vdots & 0 & \ddots & \ddots & q & 0 \\ \vdots & \vdots & \ddots & \ddots & p+q & q \\ 0 & 0 & \cdots & 0 & p & p+q \end{pmatrix}
$$

1. Obtain the recurrence formula satisfied by the determinant of $\boldsymbol{P}_n$, $|\boldsymbol{P}_n|$.
2. Express the determinant $|\boldsymbol{P}_n|$ in terms of p, q, and n, using the recurrence formula in Question IV.1.

Problem 3

Answer the following questions concerning complex functions defined over the z-plane ($z = x + iy$), where i denotes the imaginary unit.

I. For the function $f(z) = \dfrac{z}{(z^2+1)(z-1-ia)}$, where a is a positive real number:

1. Find all the poles and respective residues of $f(z)$.
2. Using the residue theorem, calculate the definite integral

$$\int_{-\infty}^{\infty} \frac{x}{(x^2+1)(x-1-ia)} dx. \tag{1}$$

II. Consider the function $g(z) = \dfrac{z}{(z^2+1)(z-1)}$ and the closed counter-clockwise path of integration C, which consists of the upper half circle C_1 with radius R ($z = R\,e^{i\theta}$, $0 \le \theta \le \pi$), the line segment C_2 on the real axis ($z = x$, $-R \le x \le 1-r$), the lower half circle C_3 with its center at $z=1$ ($z = 1 - r\,e^{i\theta}$, $0 \le \theta \le \pi$), and the line segment C_4 on the real axis ($z = x$, $1+r \le x \le R$). Here, e denotes the base of the natural logarithm, and let $r > 0$, $r \ne \sqrt{2}$ and $R > 1 + r$. An example for the case $0 < r < 1$ is illustrated in Figure 3.1.

Answer the following questions.

1. Calculate the integral $\int_C g(z)\,dz$.
2. Using the result from Question II.1, calculate the following value

$$\lim_{\varepsilon \to +0}\left[\int_{-\infty}^{1-\varepsilon} g(x)\,dx + \int_{1+\varepsilon}^{\infty} g(x)\,dx\right]. \tag{2}$$

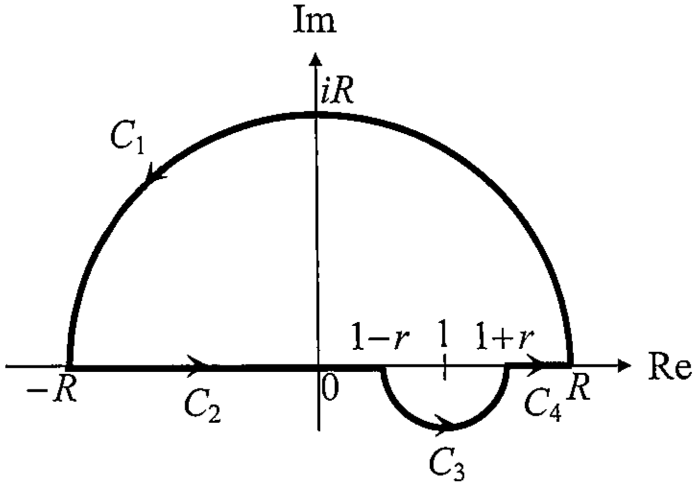

Figure 3.1

Problem 4

I. Consider surfaces presented by the following sets of equations, with parameters u and v in a three-dimensional orthogonal coordinate system xyz. Show the equations for the surfaces without the parameters and sketch them. Here, a, b, and c are non-zero real constants.

1. $x = au\cosh v, \quad y = bu\sinh v, \quad z = u^2.$
2. $x = a\dfrac{u-v}{u+v}, \quad y = b\dfrac{uv+1}{u+v}, \quad z = c\dfrac{uv-1}{u+v}.$

II. In a three-dimensional orthogonal coordinate system xyz, consider the surface S represented by the following equation, where a and b are real constants.

$$z = x^2 - 2y^2 + ax + by \qquad (1)$$

1. Determine the normal vector at a point (x, y, z) on the surface S.
2. Determine the equation for the surface T which is obtained by rotating the surface S around the z-axis by $\pi/4$. Here, the positive direction of rotation is counter-clockwise when looking at the origin from the positive side of the z-axis.
3. Consider the surface S', which is the portion of the surface S in $-1 \le x \le 1$ and $-1 \le y \le 1$. Determine the area of the projection of the surface S' onto the yz plane.
4. Calculate the length of the perimeter for the surface S' when $a = b = 0$.
5. Calculate the Gaussian curvature of the surface S at the point $\left(0, \dfrac{1}{4}, -\dfrac{1}{8}\right)$ when $a = b = 0$.

Problem 5

Let $f(t)$ be a periodic function of period T, $f(t+T)=f(t)$ $(T>0)$, and be expanded in the complex Fourier series as follows:

$$f(t)=\sum_{n=-\infty}^{\infty} F_n \exp(-i\,\omega_n t)\,. \tag{1}$$

Here, i is the imaginary unit, and t is a real number. Answer the following questions.

I. Express ω_n using T and n.

II. Let M be a positive-integer constant and $\delta(t)$ be the delta function, and define

$$\hat{f}(t)=\sum_{m=0}^{M-1} f(t)\,\delta(t-m\Delta t)\,, \qquad \Delta t=\frac{T}{M}\,. \tag{2}$$

Express

$$\lim_{\varepsilon\to+0}\int_{-\varepsilon}^{T-\varepsilon} \hat{f}(t)\exp(i\,\omega_k t)\,dt \tag{3}$$

using F_n $(n=-\infty,\cdots,-1,\,0,\,1,\cdots,\infty)$. Here k is an arbitrary integer.

III. Δt is given in Question II. Express F_j $(j=0,1,2,\cdots,M-1)$ using

$$f(0),\ f(\Delta t),\ f(2\Delta t),\cdots,\ f((M-1)\Delta t),$$

when $F_n=0$ $(n<0$ or $n\geq M)$.

IV. Calculate F_j in Question III when $f(l\Delta t)=(-1)^l$ $(l=0,1,2,\cdots,M-1)$.

Problem 6

There are n children queuing in a line. You have m candies and will begin handing out 1 or 2 candies to each child, starting from the first child in the line. You hand out the candies until reaching the end of the line or until there are no candies left. Answer the following questions. Note that n and m are positive integers.

I. Show the number of distribution patterns of candies if $n = m = 4$.

II. Show the number of distribution patterns of candies if $m \geq 2n$.

III. Define X_m as the number of distribution patterns of candies if $n \geq m$. Show the recurrence formula satisfied by X_m.

IV. Obtain X_m using the recurrence formula in Question III.

V. Consider the situation where the number of children is larger than the number of candies. Define $P(m)$ as the ratio of the number of distribution patterns (where the distribution finishes by giving 2 candies) to the total number of distribution patterns. $P(m)$ converges as m increases. Compute the convergence value.

VI. Consider the situation where $m \geq 2n$. The following rules are added to the way of handing out the candies: For the first child in the line, the probability of receiving 1 candy is $1/2$ and the probability of receiving 2 candies is $1/2$. If a child receives 1 candy, the probability of the next child receiving 1 candy is $1/2$ and the probability of receiving 2 candies is $1/2$. If a child receives 2 candies, the probability of the next child receiving 1 candy is $3/4$ and the probability of receiving 2 candies is $1/4$. Compute the probability that the n-th child in the line receives 2 candies.