2011
The Graduate School Entrance Examination
Mathematics
1:00 pm－3:30 pm

GENERAL INSTRUCTIONS

Answers should be written in Japanese or English.

1. Do not open the problem booklets whether English or Japanese until the start of the examination is announced.

2. Notify if you find any page missing, out of order or unclear.

3. Answer three problems out of the six problems of the problem booklet.

4. You are given three answer sheets. Use one answer sheet for each problem. You may use the reverse side if necessary.

5. Print your examinee number and the problem number in the designated places at the top of each answer sheet. The wedge-shaped marks on the top edge of the answer sheet represent the problem number you answer (P 1, P 2,..., P 6) on that sheet and also the class of master's course (M) or doctoral course (D) applicants. At the end of the examination, follow your proctor's instructions and cut out carefully the two corresponding wedge marks per sheet with a pair of scissors.

6. You may use the blank sheets of the problem booklet as working space and for draft solutions, but you must not detach them.

7. Any answer sheet with marks or symbols irrelevant to your answers will be considered invalid.

8. You may not take the booklet or answer sheets with you after the examination.

Examinee Number	No.

Write your examinee number in the space provided.

Problem 1

I. When n is a natural number, the indefinite integral I_n is defined as,

$$I_n = \int \frac{1}{\left(x^2 + a^2\right)^n} dx. \tag{1}$$

Here, a is a constant real number, and not equal to zero. Answer the following questions.

1. Express I_{n+1} as a recurrence equation with I_n.
2. Derive I_1 and I_2 respectively. Omit the constant of integration.
3. Calculate the following indefinite integral. Omit the constant of integration.

$$\int \frac{4x^4 + 2x^3 + 10x^2 + 3x + 9}{(x+1)\left(x^2+2\right)^2} dx. \tag{2}$$

II. Consider the following partial differential equation of $u(x,t)$, $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$. Here, c is a positive constant. Answer the following questions.

1. Making use of independent variables $\xi = x + ct$ and $\eta = x - ct$, show $\dfrac{\partial^2 u}{\partial \xi \partial \eta} = 0$. Then show that using arbitrary functions ϕ and φ, the general solution of this partial differential equation can be expressed as Eq. (3),

$$u(x,t) = \phi(x+ct) + \varphi(x-ct). \tag{3}$$

2. In the general solution derived in Question II. 1, when initial conditions are defined by Eq. (4), show that the solution can be expressed as Eq. (5).

$$u(x,0) = f(x), \quad \frac{\partial}{\partial t} u(x,t)\Big|_{t=0} = g(x). \tag{4}$$

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s) ds. \tag{5}$$

3. When initial conditions are defined by Eq. (6), obtain the solution, then draw a schematic and explain the behavior of $u(x,t)$ at $t \geq 0$. Here, c_0 is a positive constant, and $\delta(x)$ is the Delta function.

$$u(x,0) = 0, \quad \left. \frac{\partial}{\partial t} u(x,t) \right|_{t=0} = c_0 \delta(x). \tag{6}$$

Problem 2

Let A be an $n \times n$ symmetric matrix whose elements are real numbers, and $B = \begin{pmatrix} 6 & 2 & 2 \\ 2 & 7 & 0 \\ 2 & 0 & 5 \end{pmatrix}$. Answer the following questions.

I. Show that solving $Ax = b$ for x is equivalent to obtaining the stationary value of $\frac{1}{2}x^{\mathrm{T}}Ax - x^{\mathrm{T}}b$. Here, x and b are n-dimensional real vectors, A and b are known, x^{T} means transpose of x.

II. In the case of $C = D^{-1}BD = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix}$, obtain the diagonal matrix C and the 3×3 orthogonal matrix D. Here, $c_1 \geq c_2 \geq c_3$, and the elements of the first row of matrix D are positive values.

III. Obtain $\lim_{k \to \infty} \frac{a_k}{b_k}$, where k is a positive integer and $\begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix} = B^k \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$.

IV. Function $f(y_1, y_2, y_3)$ is given by Eq. (1)

$$f(y_1, y_2, y_3) = \frac{7y_1^2 + 8y_2^2 + 6y_3^2 + 4y_1(y_2 + y_3)}{4y_1^2 + 4y_2^2 + 4y_3^2}, \qquad (1)$$

where y_1, y_2, y_3 are real numbers, and $y_1^2 + y_2^2 + y_3^2 \neq 0$. Express $f(y_1, y_2, y_3)$ using $\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$, and find the minimum value of $f(y_1, y_2, y_3)$. Here $\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$ is defined by $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = D \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$, where D is obtained in Question II.

Problem 3

Consider complex functions $f(z)=\dfrac{\log(z+i)}{z^2+1}$ and $g(z)=\dfrac{\log(z-i)}{z^2+1}$, where $\log z=\log_e|z|+i\mathrm{Arg}(z)+2mi\pi$ ($-\pi<\mathrm{Arg}(z)\le\pi$, $m=0,\ \pm1,\ \pm2,\ldots$), i is the imaginary unit, and e is the base of natural logarithm. Answer the following questions.

I. Consider a counterclockwise integral contour C consisting of the upper semicircle C_1 and the diameter C_2 with radius $R\,(R>1)$ on the complex plane as shown in Figure 3.1. Obtain poles and residues of $f(z)$ in the area bounded by the contour C.

II. Evaluate $\int_{\mathrm{C}} f(z)dz$.

III. Show the following inequality on the integral contour C_1 (upper semicircle) where $z=Re^{i\theta}$, $0\le\theta\le\pi$, and $R>1$:

$$\left|\int_{\mathrm{C}_1}\frac{\log(z+i)}{z^2+i}dz\right|<\pi R\frac{\log_e(R+1)+\dfrac{3\pi}{2}}{R^2-1}. \qquad (1)$$

Use $\left|\log(Re^{i\theta}+i)\right|<\log_e(R+1)+\dfrac{3\pi}{2}$, if necessary.

IV. Consider a counterclockwise integral contour Γ consisting of the lower semicircle Γ_1 and the diameter Γ_2 with radius $R\,(R>1)$ as shown in Figure 3.2. Evaluate $\int_{\Gamma} g(z)dz$.

V. Prove the following equation:

$$\int_0^{\infty}\frac{\log_e(x^2+1)}{x^2+1}dx=\pi\log_e 2. \qquad (2)$$

If necessary, use $\int_{\mathrm{C}_1} f(z)dz\xrightarrow[R\to\infty]{}0$ and $\int_{\Gamma_1} g(z)dz\xrightarrow[R\to\infty]{}0$.

VI. Evaluate the following definite integral by using the result of Question V:

$$I = \int_0^{\frac{\pi}{2}} \log_e(\cos\theta)d\theta. \quad (3)$$

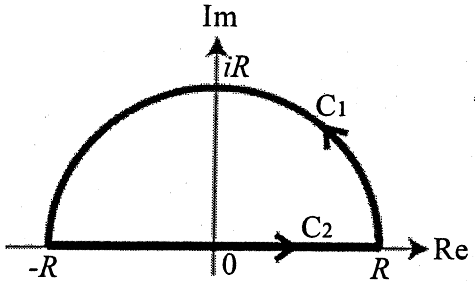

Figure 3.1

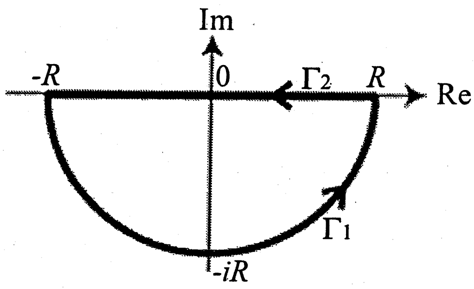

Figure 3.2

Problem 4

Points $\mathrm{A}(-1,0)$, $\mathrm{B}(1,0)$ and $\mathrm{P}(x,y)$ exist on the xy plane. Curve C is defined as a locus of point P where the product of the distances $\overline{\mathrm{AP}}$ and $\overline{\mathrm{BP}}$ is equal to a constant value $s\ (s>0)$. Answer the following questions.

I. Draw a schematic of curve C for the cases of $s=\frac{5}{4}$ and $s=\frac{3}{4}$, respectively.

II. Find the maximum value of y on curve C as a function of s.

III. Consider region D which is surrounded by curve C for the case of $s=1$ in $x \geq 0$.

1. Line $x=\sqrt{3}y$ divides region D into two domains. Obtain the area of each domain.
2. Consider the body of rotation of region D around the x axis. Find its surface area.

Problem 5

The Laplace transform $F(s)=L[f(t)]$ of function $f(t)$ is defined as

$$F(s)=\int_0^\infty f(t)e^{-st}dt, \tag{1}$$

where s is a complex number, t is a real number and $t \geq 0$.

I. Function $u(t)$ with a positive real number a is defined as

$$u(t-a)=\begin{cases}0 & t\leq a\\ 1 & t>a.\end{cases} \tag{2}$$

1. Calculate $L[u(t-a)]$.
2. Show that $L[f(t-a)u(t-a)]=e^{-as}F(s)$.
3. Solve the following differential equation using Laplace transformation.

$$\frac{d^2x(t)}{dt^2}+4\frac{dx(t)}{dt}+3x(t)=u(t-2)-u(t-5), \tag{3}$$

where $x=0$ and $\frac{dx}{dt}=0$ at $t=0$.

II. Function $g(t)$ is given for $0\leq t\leq T$. Let $g(0)=g(T)=0$. Function $h(t)$ ($t\geq 0$) is defined as

$$h(t)=(-1)^n g(t-nT), \tag{4}$$

where n is an integer which satisfies $0\leq t-nT\leq T$.
Then the Laplace transform of $h(t)$ can be expressed using function $A(s)$ as

$$L[h(t)]=A(s)\int_0^T g(t)e^{-st}dt. \tag{5}$$

Determine $A(s)$.

Problem 6

The number of ants N in a colony is estimated by two successive captures. At the first capture, the number of ants captured was n_1 and all the ants captured were marked and released. At the second capture, the number of ants captured was n_2 and included $m\,(m \neq 0)$ marked ants. Assume that two captures were unintentional samplings from the same population of ants, and the capture and marking do not affect the behavior of the ants.

I. Find the number of combination where the number of marked ants included in the second capture is m.

II. Express $P_m(N)$, the probability of finding m marked ants in the second capture with N, n_1, n_2 and m.

III. Find the condition on N for satisfying $P_m(N) \geq P_m(N+1)$.

IV. It is known that $P_m(N)$ takes a maximum value with respect to N. For N that gives a maximum value of $P_m(N)$, express N with n_1, n_2 and m.