2010
The Graduate School Entrance Examination
Physics
9:00－11:00 am

GENERAL INSTRUCTIONS

Answers should be written in Japanese or English.

1. Do not open the problem booklets whether English or Japanese until the start of the examination is announced.

2. Notify if you find any page missing, out of order or unclear.

3. Answer two problems out of the four problems of the problem booklet.

4. You are given two answer sheets. Use one answer sheet for each problem. You may use the reverse side if necessary.

5. Print your examinee number and the problem number in the designated places at the top of each answer sheet. The wedge-shaped marks on the top edge of the answer sheet represent the problem number you answer on that sheet and also the class of master's course (M) and doctoral course (D) applicants. At the end of the examination, follow your proctor's instructions and cut out carefully the two corresponding wedge marks per sheet with a pair of scissors.

6. You may use the blank sheets of the problem booklet as working space and for draft solutions, but you must not detach them.

7. Any answer sheet with marks or symbols irrelevant to your answers will be considered invalid.

8. You may not take the booklet or answer sheets with you after the examination.

Examinee Number	No.

Write your examinee number in the space provided.

Problem 1

As shown in Figure 1.1, a slope with a uniform gap is made by placing two identical triangular plates perpendicular to the horizontal plane and by parallel translation of one of them in the direction normal to the plates. A circular disk having an eccentric shaft, fixed to the disk in its normal direction, is considered. The disk is inserted parallel to the two plates with its shaft on the slope. There is no friction between the slope and the shaft. The disk touches the plates only by the shaft. Now, as shown in Figure 1.2, an inertial frame (x, y) is defined by the x axis taken horizontally in the right direction and the y axis vertically upward. Moreover, a moving frame (x', y'), which moves with the eccentric shaft (point A), is considered with the x' axis taken in the descent direction and the y' axis in the direction normal outward to the slope. The two triangular plates are fixed to the horizontal plane.

Generally, the disk falls along the slope oscillating around the shaft under a uniform gravity. Let us, then, consider a special situation where the falling disk is stationary relative to the frame (x', y'). As shown in Figure 1.2, let the angle of the slope to the horizontal plane be denoted by θ $(0 < \theta < \pi/2)$, the radius and the mass of the disk by a and m, respectively, the distance between point A and the center of gravity O of the disk by h, and the acceleration of gravity by g. Answer the following questions, assuming that the diameter and the mass of the shaft are negligible.

I. When R denotes the magnitude of the constraint force which the shaft receives from the slope, write equations of motion in the x and y directions for the center of gravity O relative to the inertial frame (x, y).

II. Obtain the x-component and the y-component of the acceleration $\boldsymbol{G}$ of the point A relative to the frame (x, y).

III. Obtain the magnitude of the constraint force R acting on point A.

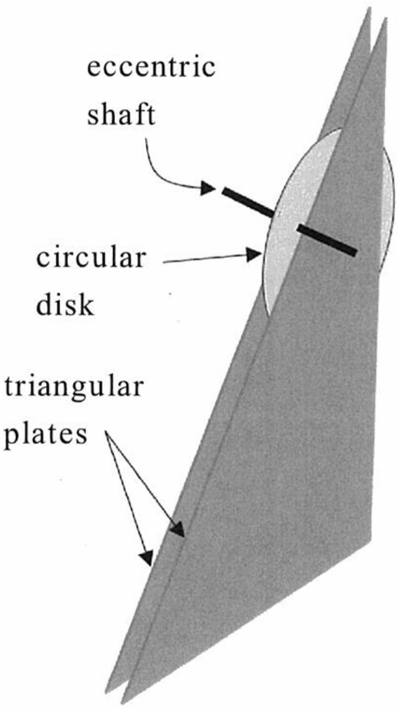

Figure 1.1

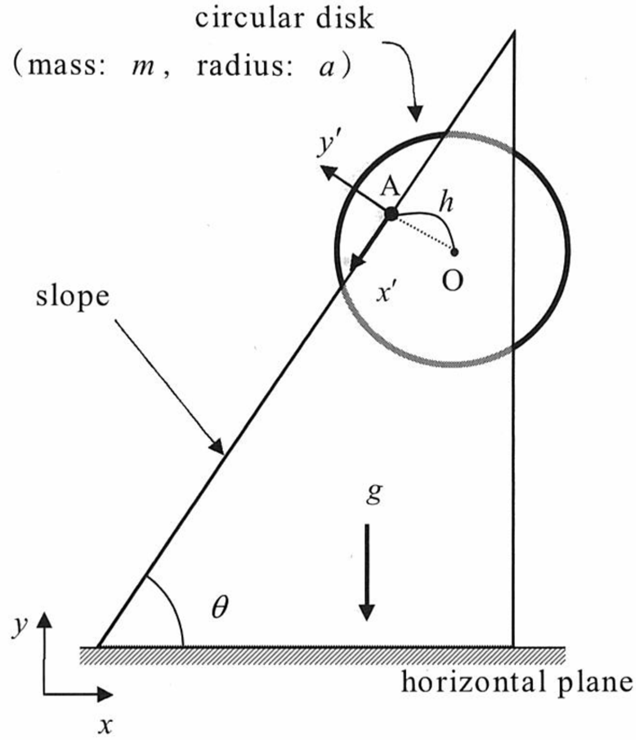

Figure 1.2

IV. Obtain the moment of inertia I of the disk around the shaft at point A.

Now let us consider the case where the amplitude of oscillation around the shaft at point A is small.

V. When φ denotes the angular displacement from the center of oscillation, obtain the period T of the oscillation, assuming that the perturbation of the acceleration of point A due to the oscillation can be neglected. Use I to denote the moment of inertia of the disk around the shaft at point A.

VI. When the distance h is given by $h = a(1-\cos\theta)$, obtain the angle θ at which the period becomes minimum.

Problem 2

Consider the attenuation of an electromagnetic wave in an electric conductor (electrical conductivity σ, dielectric constant ε, magnetic permeability μ). As shown in Fig. 2.1, an electromagnetic plane wave (angular frequency ω, time t) with an electric field (E_x) along the x axis and a magnetic field (H_y) along the y axis propagates along the z axis from vacuum to the conductor surface (infinite plane, $z=0$) with vertical incidence. Using the Maxwell's equations (magnetic field $\boldsymbol{H}$, electric field $\boldsymbol{E}$, current density $\boldsymbol{J}$, electric flux density $\boldsymbol{D}$, magnetic flux density $\boldsymbol{B}$) expressed by Eqs. (1) and (2), answer the following questions about the electromagnetic wave in a conductor ($z>0$). Here, σ, ε and μ are constants (real numbers) that are independent of ω, and i is the imaginary unit.

$$\mathrm{rot}\,\boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \qquad (1)$$

$$\mathrm{rot}\,\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad (2)$$

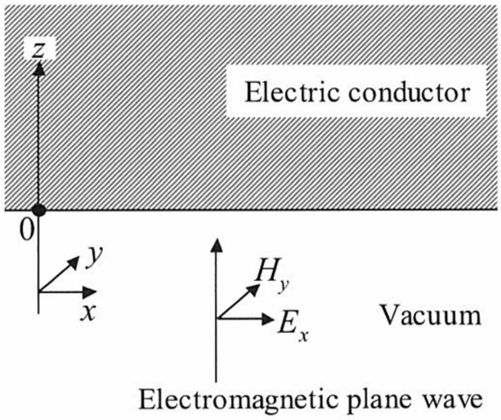

Figure 2.1

I. Derive two equations showing the relation between E_x and H_y.

II. Setting $E_x = E(z)\, e^{i\omega t}$ and $H_y = H(z)\, e^{i\omega t}$, the equations of $E(z)$ and $H(z)$ are expressed by the following equations.

$$\frac{\partial^2 E(z)}{\partial z^2} = (-\varepsilon\mu\omega^2 + i\mu\sigma\omega)E(z) \qquad (3)$$

$$\frac{\partial^2 H(z)}{\partial z^2} = (-\varepsilon\mu\omega^2 + i\mu\sigma\omega)H(z) \qquad (4)$$

Eqs. (3) and (4) are assumed to be $\partial^2 E(z)/\partial z^2 = p^2 E(z)$ and $\partial^2 H(z)/\partial z^2 = p^2 H(z)$, respectively. Setting $p = \alpha + i\beta$, we obtain the following equations with respect to α and β.

$$\alpha = \omega \left[\frac{\varepsilon\mu}{2} \left(\sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega} \right)^2} - 1 \right) \right]^{1/2} \tag{5}$$

$$\beta = \omega \left[\frac{\varepsilon\mu}{2} \left(\sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega} \right)^2} + 1 \right) \right]^{1/2} \tag{6}$$

Setting $E(z) = E_0 e^{-pz}$, answer the following questions. Here, E_0 is a constant (real number).

1. Express E_x using E_0, α, β, and other variables.
2. The phase difference between E_x and H_y is set to be γ. Show H_y using E_0, α, β, γ, and other variables. In addition, obtain γ.

III. Using the results of Question II, express the ratio of the magnetic field energy (U_H) and the electric field energy (U_E), U_H / U_E, using σ, ε and ω. In addition, the energy ratio, U_H / U_E, differs for vacuum and a conductor ($\sigma / \varepsilon\omega >> 1$). Explain this in about; 50 words in English, or 100 characters in Japanese.

IV. Setting $l = \sqrt{\dfrac{2}{\mu\sigma\omega}}$, derive an approximate expression of $E(z)$ in a conductor under the condition of $\sigma / \varepsilon\omega >> 1$ using E_0, l, z, and other variables. In addition, derive an approximate expression of $H(z)$ in this conductor using E_0, σ, l, z, and other variables. Furthermore, sketch the real part of $E(z)$ as a function of z / l in this conductor.

V. Derive an approximate expression of $E(z)$ in a material under the condition of $\sigma / \varepsilon\omega << 1$ using E_0, σ, μ, ε, ω, z, and other variables. In addition, derive an approximate expression of $H(z)$ in this material using E_0, σ, μ, ε, ω, z, and other variables. Furthermore, sketch the real part of $E(z)$ as a function of z in this material.

Problem 3

Consider the cooling of a gas by the Joule-Thomson effect.

I. As shown in Figure. 3.1, a porous plug is fixed in an adiabatic container. Initially, a gas of n mol occupies volume V_1 in Region 1, which is on the left side of the porous plug. Pressure P_1 is applied by Piston 1 from the left side of Region 1. On the right side of the porous plug, pressure P_2 is applied by Piston 2. When $P_1 > P_2$, the gas flows gradually from Region 1 to Region 2 through the porous plug and finally occupies volume V_2 in Region 2 as shown in Figure 3.2. Show that this process is a constant enthalpy process.

II. Show that a decrease in the pressure P results in an increase in the entropy S when the enthalpy H remains constant.

The Joule-Thomson coefficient μ_{JT} can be expressed as,

$$\mu_{\mathrm{JT}} = \left(\frac{\partial T}{\partial P}\right)_H = \frac{1}{C_P}\left\{T\left(\frac{\partial V}{\partial T}\right)_P - V\right\}, \qquad (1)$$

where T is the absolute temperature, V is the volume, and C_P is the specific heat capacity at constant pressure.

III. Show that the Joule-Thomson coefficient for an ideal gas is zero and explain the reason from the property of the ideal gas.

IV. Consider the Joule-Thomson coefficient for a non-ideal gas which obeys the Berthelot equation of state $\{P + an^2/(TV^2)\}(V - nb) = nRT$, where a and b are constants that depend on the gas. R is the gas constant.

1. Obtain $(\partial P/\partial T)_V$ and $(\partial P/\partial V)_T$.
2. Show that the Joule-Thomson coefficient can be approximated as,

$$\mu_{\mathrm{JT}} \approx \frac{n}{C_P}\left(\frac{3a}{RT^2} - b\right). \tag{2}$$

Assume $na << RT^2V$ and $nb << V$ for a dilute gas.

V. Obtain the critical temperature T_c of the gas which obeys the Berthelot equation of state. Also write down the inversion temperature T_i in terms of T_c, where T_i gives $\mu_{\mathrm{JT}} = 0$ in Eq. (2).

VI. Show and explain the temperature range, where the cooling occurs by the Joule-Thomson effect, using Eq. (2).

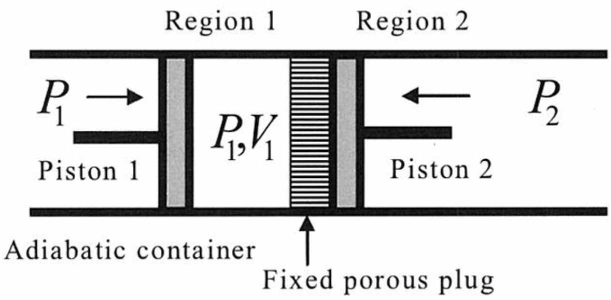

Figure 3.1

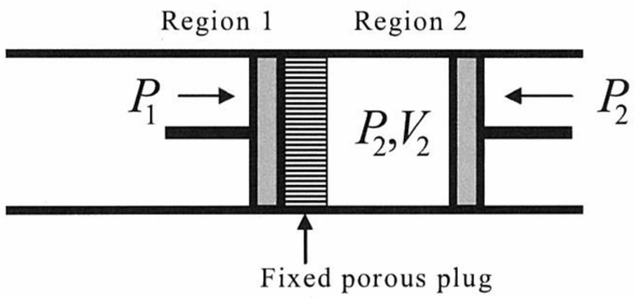

Figure 3.2

Problem 4

Consider a hydrogen atom. An electron of charge $-e$ exists around a proton of charge $+e$. The volume of these two particles are assumed to be zero. The mass of the proton is large enough compared to that of the electron, and we assume that the position of the proton is fixed. Now, the origin of the Cartesian coordinate system is set at the position of the proton, and the position of the electron is $\boldsymbol{r} = (x, y, z)$ and the distance from the origin to the electron is $r \equiv |\boldsymbol{r}|$. Here $\hbar$ is the Planck's constant divided by 2π and i is the imaginary unit.

I. The Coulomb force between the proton and the electron is,

$$F = \frac{e^2}{4\pi\varepsilon_0 r^2}, \tag{1}$$

where ε_0 is the dielectric constant in vacuum. Derive the potential energy V when the electron is at distance r, from the proton. Set the constant to satisfy the condition, $V \to 0\ (r \to \infty)$.

II. When the system is in a steady state, show the Schrödinger equation for the electron. Here the wave function of the electron is φ, and the eigenvalue of the energy of the electron is E.

III. Change the coordinate system from the Cartesian coordinate system (x, y, z) to the polar coordinate system (r, θ, ϕ) as,

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta\,. \end{cases} \tag{2}$$

Draw a figure of (x, y, z) coordinate system as shown in Figure 4.1 and depict the relation between the (r, θ, ϕ) and the (x, y, z) coordinate systems. Add appropreate lines and variables.

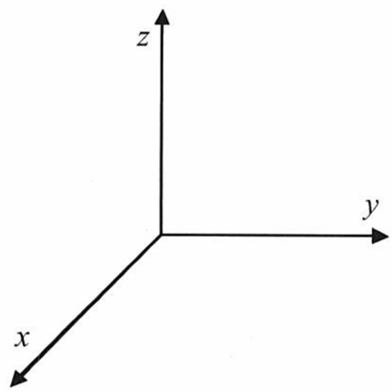

Fig. 4.1

Now consider to represent the Schrödinger equation that you have derived in Question II to a polar coordinate system (r,θ,ϕ) defined as Eq. (2). Here, we will try the separation of variables on the wave function for the electron represented in the polar coordinate system, $\varphi(r,\theta,\phi)$ as,

$$\varphi(r,\theta,\phi) = R(r)Y(\theta,\phi). \tag{3}$$

Then, $Y(\theta,\phi)$ should satisfy the equation,

$$\hat{\Lambda}Y(\theta,\phi) = CY(\theta,\phi), \tag{4}$$

where C is a constant and $\hat{\Lambda}$ is an operator defined as,

$$\hat{\Lambda} \equiv \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \tag{5}$$

It is known that there exists a set of orthonormal functions $Y_l^m(\theta,\phi)$ as a solution of Eq. (4), which satisfies the following equations,

$$\hat{\Lambda}Y_l^m(\theta,\phi) = -l(l+1)Y_l^m(\theta,\phi) \tag{6}$$

$$\frac{\partial}{\partial\phi}Y_l^m(\theta,\phi) = imY_l^m(\theta,\phi). \tag{7}$$

IV. Here, we define an angular momentum operator around the origin of the axes as,

$$\hat{\boldsymbol{l}} = \left(\hat{l}_x, \hat{l}_y, \hat{l}_z\right). \tag{8}$$

It is known that the operator defined as,

$$\hat{\boldsymbol{l}}^2 \equiv \hat{l}_x^{\,2} + \hat{l}_y^{\,2} + \hat{l}_z^{\,2}, \tag{9}$$

is represented in the polar coordinate (r, θ, ϕ) as,

$$\hat{\boldsymbol{l}}^2 = -\hbar^2 \hat{\Lambda}. \tag{10}$$

Derive the expectation value $\langle \boldsymbol{l}^2 \rangle$, where $\boldsymbol{l}^2$ is the square sum of the angular momentum for an electron with the wave function $\varphi(r, \theta, \phi)$.

V. Derive the expectation value of $\left(\boldsymbol{l}^2 - \langle \boldsymbol{l}^2 \rangle\right)^2$ for the electron whose wave function is $\varphi(r, \theta, \phi)$.

VI. When we measure $\boldsymbol{l}^2$ experimentally, what kind of results should be expected, judging from the solution of Question V?

VII. It is known that the angular momentum around the z-axis is represented in the polar coordinate as,

$$\hat{l}_z = -i\hbar \frac{\partial}{\partial \phi}. \tag{11}$$

Then, what kind of relationship exists between measured values of l_z and $\boldsymbol{l}^2$? Explain briefly.