2010
The Graduate School Entrance Examination
Mathematics
1:00－3:30 pm

GENERAL INSTRUCTIONS

Answers should be written in Japanese or English.

1. Do not open the problem booklets whether English or Japanese until the start of the examination is announced.

2. Notify if you find any page missing, out of order or unclear.

3. Answer three problems out of the six problems of the problem booklet.

4. You are given three answer sheets. Use one answer sheet for each problem. You may use the reverse side if necessary.

5. Print your examinee number and the problem number in the designated places at the top of each answer sheet. The wedge-shaped marks on the top edge of the answer sheet represent the problem number you answer on that sheet and also the class of master's course (M) or doctoral course (D) applicants. At the end of the examination, follow your proctor's instructions and cut out carefully the two corresponding wedge marks per sheet with a pair of scissors.

6. You may use the blank sheets of the problem booklet as working space and for draft solutions, but you must not detach them.

7. Any answer sheet with marks or symbols irrelevant to your answers will be considered invalid.

8. You may not take the booklet or answer sheets with you after the examination.

Examinee Number	No.

Write your examinee number in the space provided.

Problem 1

I. Suppose that the following simultaneous differential equations,

$$\frac{dx}{dt} = f(x,y,t), \tag{1}$$

$$\frac{dy}{dt} = g(x,y,t), \tag{2}$$

are solved numerically under the initial conditions $x(t_0) = x_0$, $y(t_0) = y_0$ and that numerical solutions are defined as $x_0, x_1, \ldots, x_i, x_{i+1}, \ldots$ and $y_0, y_1, \ldots, y_i, y_{i+1}, \ldots$ corresponding to $t_0, t_1, \ldots, t_i, t_{i+1}, \ldots$ which are values of a free variable t with a step size h. Supposing the second- and higher- order terms in Taylor's expansion of $x(t_i + h)$ around t_i can be neglected in Eq. (1), express x_{i+1} with h, x_i, and $f(x_i, y_i, t_i)$.

II. Consider the simultaneous differential equations,

$$\frac{dx}{dt} = x\left(1 - \sqrt{x^2 + y^2}\right) - y, \tag{3}$$

$$\frac{dy}{dt} = x + y\left(1 - \sqrt{x^2 + y^2}\right). \tag{4}$$

Obtain values of x_1 in the cases of $h = 0.1$ and $h = 0.5$ under the initial conditions of $x(0) = 2$, $y(0) = 0$ applying the result of Question I to Eq. (3). Then, show that the errors increase as the step size becomes larger. Here, the values of analytical solution are $x(0.1) = 1.82$, $x(0.5) = 1.26$ to the second decimal place.

III. Introduce $x = r\cos\theta$, $y = r\sin\theta$. Then, derive simultaneous differential equations from Eqs. (3) and (4) using variables r and θ.

IV. Find general solutions of the differential equations which are obtained in Question III under the following conditions: $r = r_0$ $(r_0 > 0)$ and $\theta = 0$ when $t = 0$.

V. Draw solution curves to Eqs. (3) and (4) on the *xy* plane for $t \geq 0$ in the following cases: i) $0 < r_0 < 1$ and, ii) $r_0 > 1$.

Problem 2

Consider the following differential equation,

$$\frac{d}{dt}\boldsymbol{p} = \boldsymbol{A}\boldsymbol{p}, \tag{1}$$

where $\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$, $\boldsymbol{p} = \begin{bmatrix} X(t) \\ Y(t) \\ Z(t) \end{bmatrix}$. Answer the following questions.

I. For matrix $\boldsymbol{A}$, find all the eigenvalues and the corresponding eigenvectors of norm 1, within complex numbers.

II. Among the eigenvectors of norm 1 of matrix $\boldsymbol{A}$, let $\boldsymbol{u}_1$ be a real vector, whose elements are real numbers. Let $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$ be real vectors that construct an orthonormal basis with $\boldsymbol{u}_1$. Define a square matrix $\boldsymbol{T}$ of order 3 as $\boldsymbol{T} = [\boldsymbol{u}_1 \quad \boldsymbol{u}_2 \quad \boldsymbol{u}_3]$ by aligning the three vectors $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$. Obtain one example of matrix $\boldsymbol{T}$.

III. Vector $\boldsymbol{q}$, defined as $\boldsymbol{q} = \boldsymbol{T}^{-1}\boldsymbol{p}$, satisfies the following differential equation,

$$\frac{d}{dt}\boldsymbol{q} = \boldsymbol{B}\boldsymbol{q}. \tag{2}$$

Derive matrix $\boldsymbol{B}$ in Eq. (2) using matrix $\boldsymbol{T}$ obtained in Question II.

IV. Let $\boldsymbol{q}_0$ be the initial value of $\boldsymbol{q}$ at $t = 0$. Then the solution of Eq. (2) is expressed as $\boldsymbol{q} = \exp(\boldsymbol{B}t)\boldsymbol{q}_0$. Find $\exp(\boldsymbol{B}t)$ for matrix $\boldsymbol{B}$ derived in Question III.

For an arbitrary square matrix $\boldsymbol{X}$, $\exp(\boldsymbol{X})$ is defined as

$$\exp(\boldsymbol{X}) = \sum_{n=0}^{\infty} \frac{1}{n!}\boldsymbol{X}^n \quad .$$

V. Consider a point $P(X(t), Y(t), Z(t))$, whose position vector is vector $\boldsymbol{p}$, in the xyz space. Let $\boldsymbol{p}_0$ be the initial value of $\boldsymbol{p}$ at $t = 0$. When $\boldsymbol{p}_0 = \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \end{bmatrix}$, point P moves on a cylindrical surface. Derive an equation to describe this cylindrical surface.

Problem 3

I. Consider $I(x) = \sum_{n=-\infty}^{\infty} g_n(x)$ and $g_n(x) = \dfrac{1}{i(2n+1)\pi - x}$, where x is a real variable and i is the imaginary unit.

1. Consider a real function $I_k(x) = g_k(x) + g_{-k-1}(x)$, where k is a positive integer. Plot the graph of $I_k(x)$, and indicate x where local maximum and local minimum values are taken.
2. Show that $I(x)$ can be expressed as $I(x) = \int_{\mathrm{C}} f(z)\dfrac{1}{z-x}dz$ by choosing an appropriate integral contour C on the complex plane and introducing a complex function $f(z) = \dfrac{1}{2\pi i}\left(\dfrac{1}{\exp(z)+1}\right)$, where z is an imaginary number.
3. Using the result of Question 2, evaluate $I(x)$ and plot its graph.

II. Consider $J(x) = \lim_{y \to x} \sum_{n=-\infty}^{\infty} \left(\dfrac{1}{i(2n+1)\pi - x} \times \dfrac{1}{i(2n+1)\pi - y} \right)$, where x and y are real numbers. Evaluate $J(x)$ and plot its graph.

Problem 4

Consider curve C on the xy plane defined by

$$4\left(x^2+y^2\right)^2-\left(x^2-y^2\right)=0. \quad (1)$$

Answer the following questions.

I. Find the coordinates of the point(s) on curve C where y takes local maximum value(s).

II. Draw a schematic of curve C.

III. Obtain the area surrounded by curve C.

IV. Consider the body of rotation of curve C around the x axis. Obtain its surface area.

V. Obtain the total length of curve C. If necessary, use $\int_0^1 \frac{du}{\sqrt{1-u^4}} \cong 1.31$.

Problem 5

The Fourier transform $F(\omega)$ of a function $f(x)$ is defined by

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp(-i\omega x) dx, \tag{1}$$

where i is the imaginary unit. Then the Fourier inverse transform is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) \exp(i\omega x) d\omega. \tag{2}$$

Answer the following questions.

I. Calculate the Fourier transform $F(\omega)$ for a Gaussian function $f(x) = \exp\left(-\frac{x^2}{a^2}\right)$. If necessary, use $\int_{-\infty}^{\infty} \exp(-\alpha x^2) dx = \sqrt{\frac{\pi}{\alpha}}$.

II. The convolution of two functions $f(x)$ and $g(x)$ is defined by

$$h(x) = \int_{-\infty}^{\infty} f(y) g(x-y) dy. \tag{3}$$

Show that

$$H(\omega) = \sqrt{2\pi}\, F(\omega) G(\omega), \tag{4}$$

where $F(\omega)$, $G(\omega)$, and $H(\omega)$ are Fourier transforms of $f(x)$, $g(x)$, and $h(x)$.

III. Determine function $f(x)$, a solution of the integral equation

$$\int_{-\infty}^{\infty} f(y) \exp\left(-\frac{(x-y)^2}{b^2}\right) dy = \exp\left(-\frac{x^2}{a^2}\right), \tag{5}$$

where $a > b > 0$.

Problem 6

Consider a continuous random variable X, whose expected value $E(X)$ and variance $V(X)$ are known to be 0 and 10, respectively. Now we consider estimating the value of X from obtained measurements. Answer the following questions. If necessary, use Eqs. (1) and (2).

$$V(\alpha X+\beta Y)=\alpha^2 V(X)+\beta^2 V(Y)+2\alpha\beta\,\mathrm{Cov}(X,Y), \quad (1)$$
$$\mathrm{Cov}(X,Y)=E(XY)-E(X)E(Y), \quad (2)$$

where $\mathrm{Cov}(X,Y)$ is the covariance between X and Y, and α and β are arbitrary constants.

I. Consider estimating the value of X from a measurement Y, where Y is the sum of X and the error W, i.e., $Y=X+W$. Assume $E(W)=0$, $V(W)=1$, and $E(XW)=0$.

1. Find $E(Y)$ and $V(Y)$.
2. Suppose we obtain an estimate Z in the form of $Z=aY$, where a is a constant. Determine the value of a so that the expected value of the squared difference between Z and X, i.e., $E((Z-X)^2)$ is minimized.

II. Consider estimating the value of X from N number of measurements $Y_1, Y_2, \ldots, Y_N$. Assume $Y_i = X + W_i$, $E(W_i)=0$, $V(W_i)=1$, $E(XW_i)=0$ for $i=1,2,\ldots,N$, and $E(W_i W_j)=0$ for $i \neq j$.

1. First, suppose we obtain an estimate Z in the form of $Z=b\sum_{i=1}^{N} Y_i$, where b is a constant. Determine the value of b so that $\varepsilon = E((Z-X)^2)$ is minimized. In addition, find the minimum value of ε.
2. Next, suppose we compute the estimate Z_i by the update equation: $Z_i=(1-c_i)Z_{i-1}+c_i Y_i$, for consecutive values of Y_i where $i=1,2,\ \ldots\ ,N$. Express $\varepsilon_i = E((Z_i - X)^2)$ which is the expected value of the squared difference between Z_i and X in terms of ε_{i-1} and c_i. Note that $E(Z_{i-1}W_i)=0$.
3. In the previous question, suppose that we choose the value of each c_i so that ε_i is minimized. Express c_i and ε_i in terms of ε_{i-1}. Assume $Z_0=0$ and $\varepsilon_0 = E((Z_0-X)^2)=10$.